Representation Theory of Finite Groups
B-III Semestral Exam
Feb. 2024

Time: 3 hours

Max score: 50

Notations: G denotes a finite group throughout, and all representations are over the field of complex numbers.

Answer any 5 questions.

(1) (a) Show that a group G is abelian if and only if all irreducible representations are of degree 1.
(b) Let G be a group and H an abelian subgroup. Show that, if d is the degree of an irreducible representation of G, then $d \leq [G : H]$, where $[G : H]$ denotes the index of H in G. (4+6)

(2) (a) Write down the character table of $D_8 = \langle r, s : \ r^4 = e = s^2, rs = sr^{-1}\rangle$, the dihedral group of order 8.
(b) Let $H = \{e, r, r^2, r^3\} \subseteq D_8$. Let $\phi : H \to \mathbb{C}^*$ be the 1-dimensional representation of H such that $\phi(r^k) = i^k$. Find the decomposition of the induced representation $\mathrm{Ind}_H^{D_8}\phi$ into irreducible representations of D_8. (5+5)

(3) (a) Prove that the degree of an irreducible representation of G divides the order of G.
(b) Show that all characters of the symmetric group S_n are real. (8+2)

(4) Let V be a representation of G.
(a) Define the dual representation $V^* = \mathrm{Hom}(V, \mathbb{C})$ of G.
(b) If χ is the character of V, show that the character of V^* is $\overline{\chi}$.
(c) Show that V is an irreducible representation of G if and only if V^* is an irreducible representation of G. (2+4+4)

(5) Let G be a non-abelian group of order 21.
(a) Determine the degrees of the irreducible representations of G
(b) How many irreducible representations G has of each degree (up to equivalence)?
(b) Determine the number of conjugacy classes of G. (4+4+2)

(6) (a) Define Specht representation S^λ of the symmetric group S_n corresponding to the partition λ of n.
(b) Prove that the Specht representation corresponding to the partition $\lambda = (n-1, 1)$ of n is the standard representation of S_n.
(6+4)
